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# Hata-Yamaguti's result on Takagi function and its applications to digital sum problems

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## 1 Introduction

Let  $n \in \mathbb{N}$  and denote its binary expansion by  $n = \sum_{k \geq 0} \alpha_k(n) 2^k$  with  $\alpha_k(n) \in \{0, 1\}$ . We define

$$\begin{aligned} s(n) &= \sum_{k \geq 0} \alpha_k(n) && \text{(the binary digital sum),} \\ F(\xi, N) &= \sum_{n=0}^{N-1} e^{\xi s(n)} && \text{(the exponential sum),} \\ S_p(N) &= \sum_{n=0}^{N-1} s(n)^p && \text{(the power sum)} \end{aligned}$$

for  $N \in \mathbb{N}$  and  $p, \xi \in \mathbb{R}$ . We first review some fundamental results on these sums. If  $N$  is a power of 2, we immediately have  $F(\xi, N) = N^{\log_2(1+e^\xi)}$  and  $S_1(N) = N \log_2 N / 2$ . However it is not so easy to obtain explicit formulas for arbitrary  $N \in \mathbb{N}$ . In early times the asymptotic behavior of  $S_1(N)$  was studied:

$$\begin{aligned} S_1(N) &\sim N \frac{\log_2 N}{2} \quad (N \rightarrow \infty) && \text{(Bush [1]),} \\ S_1(N) &= N \frac{\log_2 N}{2} + O(N) \quad (N \rightarrow \infty) && \text{(Mirsky [8]).} \end{aligned}$$

Finally, Trollope [18] obtained a precise formula for  $S_1(N)$  and Delange [4] gave its elegant proof. Let  $F$  be a nowhere differentiable continuous periodic function of period 1 given by

$$F(x) = 1 - x - 2^{1-x} T\left(\frac{1}{2^{1-x}}\right), \quad 0 \leq x \leq 1$$

with  $T$  the Takagi function. Then

$$S_1(N) = N \frac{\log_2 N}{2} + \frac{N}{2} F(\log_2 N) \quad (\text{Trollope, Delange}).$$

Coquet [3] obtained a precise formula for positive integer powers.

**Theorem 1.1** (Coquet [3]) *There are periodic functions  $G_{p,\tau} : \mathbb{R} \rightarrow \mathbb{R}$ ,  $0 \leq \tau \leq p$ , of period 1, such that*

$$S_p(N) = N \left(\frac{\log_2 N}{2}\right)^p + N \sum_{0 \leq \tau < p} (\log_2 N)^\tau G_{p,\tau}(\log_2 N).$$

for every integer  $p \geq 1$ . Furthermore  $G_{p,\tau}$  verify

$$(1) \quad 2^{-\tau} \binom{d}{\tau} + \sum_{\tau < p < d} \binom{d}{p} G_{p,\tau} = 2^{1-d} \binom{d}{\tau} + \sum_{\tau < q < d} 2 \binom{q}{\tau} G_{d,q}$$

for  $d \geq 2$  and  $\tau \leq d - 2$ .

**Theorem 1.2** (Coquet [3])

$$S_2(N) = N \left( \frac{\log_2 N}{2} \right)^2 + N \frac{\log_2 N}{2} \left\{ \frac{1}{2} + F(\log_2 N) \right\} + NG(\log_2 N),$$

where  $G$  is a nowhere differentiable continuous periodic function of period 1.

An explicit form of the function  $G$  is stated in Osbaldestin [13]. However, for  $p \geq 3$ , we cannot get such an explicit formula via induction formulae (1) and the continuity of  $G_{p,\tau}$  is unknown.

Concerning  $F(\xi, N)$ , Stolarsky [17] proved that  $F(\log_2 2, N)/N^{\log_2 3}$  is not well-behaved asymptotically. Harborth [6] obtained the following estimates:

$$(2) \quad \begin{aligned} \limsup_{N \rightarrow \infty} \frac{F(\log_2 2, N)}{N^{\log_2 3}} &= 1, \\ 0.812556 &< \liminf_{N \rightarrow \infty} \frac{F(\log_2 2, N)}{N^{\log_2 3}} < 0.812557. \end{aligned}$$

We now introduce a function  $G_\xi$  by

$$G_\xi(\log_2 N) = \frac{F(\xi, N)}{N^{\log_2(1+\epsilon^\xi)}}.$$

Coquet [3] and Stein [16] investigated the properties of  $G_\xi$ . Stein proved that  $G_\xi$  is a continuous periodic function of period 1 by giving a formula of  $F$ . However, it is unknown if  $G_\xi$  is differentiable. In this note, we get a simple explicit formula of  $F(\xi, N)$  by the use of the connection between  $s(n)$  and the binomial measure  $\mu_r$ . And using the results obtained in Hata-Yamaguti [7] and Sekiguchi-Shiota [15], we derive explicit formulas of the power sum  $S_p(N)$ . We notice that the higher order derivatives of the distribution function of  $\mu_r$  with respect to  $r$  play an important role in the explicit formula of  $S_p(N)$ . The results in this note can be extended to the sum of  $q$ -adic digits by the use of multinomial measures (see M-O-S-S [9]).

## 2 Hata-Yamaguti's result

Let  $I = I_{0,0} = [0, 1]$  and

$$I_{n,j} = \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right), \quad j = 0, 1, \dots, 2^n - 2, \quad I_{n,2^n-1} = \left[ \frac{2^n-1}{2^n}, 1 \right]$$

for  $n = 1, 2, 3, \dots$

Define the binomial measure  $\mu_r$  ( $0 < r < 1$ ) by a probability measure on  $I$  such that

$$\mu_r(I_{n+1,2j}) = r\mu_r(I_{n,j}), \quad \mu_r(I_{n+1,2j+1}) = (1-r)\mu_r(I_{n,j})$$

for  $n = 0, 1, 2, \dots, j = 0, 1, \dots, 2^n - 1$ .

We denote the distribution function of  $\mu_r$  by  $L$ :

$$L(r, x) = \mu_r([0, x]).$$

It is well-known that  $L(r, \cdot)$  is a strictly increasing continuous and singular function except for  $r = 1/2$  (see Salem [14]). It immediately follows that  $L(r, \cdot)$  satisfies the system of infinitely many difference equations:

$$\begin{cases} L(r, \frac{2j+1}{2^{n+1}}) - (1-r)L(r, \frac{j}{2^n}) - rL(r, \frac{j+1}{2^n}) = 0, \\ L(r, 0) = 0, \quad L(r, 1) = 1, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

This system is equivalent to the following functional equation:

$$L(r, x) = \begin{cases} rL(r, 2x), & 0 \leq x \leq \frac{1}{2}, \\ (1-r)L(r, 2x-1) + r, & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let

$$\begin{aligned} R(x) &= 1_{I_{1,0}}(x) - 1_{I_{1,1}}(x), \quad 0 \leq x \leq 1, \\ \phi(x) &= \begin{cases} 2x & 0 \leq x < 1/2, \\ 2x-1 & 1/2 \leq x \leq 1, \end{cases} \\ \psi(x) &= \int_0^x 2R(t)dt, \quad 0 \leq x \leq 1. \end{aligned}$$

The Takagi function  $T$  is defined by

$$T(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \psi(\phi^n(x)) = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi^n(x), \quad 0 \leq x \leq 1.$$

It is well-known that  $T$  is a nowhere differentiable continuous function. And  $T$  satisfies the system of infinitely many difference equations:

$$\begin{cases} T(\frac{2j+1}{2^{n+1}}) - \frac{1}{2}T(\frac{j}{2^n}) - \frac{1}{2}T(\frac{j+1}{2^n}) = \frac{1}{2^{n+1}}, \\ T(0) = 0, \quad T(1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

Hata-Yamaguti [7] have obtained the following formula which connects the Takagi function  $T$  with the function  $L$ .

**Theorem 2.1** (Hata-Yamaguti [7]) *We have*

$$\frac{1}{2} \frac{\partial}{\partial r} L(r, x) \Big|_{r=\frac{1}{2}} = T(x).$$

**Remark 2.1** We have

$$\begin{cases} \frac{\partial}{\partial r} L(r, \frac{2j+1}{2^{n+1}}) - (1-r) \frac{\partial}{\partial r} L(r, \frac{j}{2^n}) - r \frac{\partial}{\partial r} L(r, \frac{j+1}{2^n}) = L(r, \frac{j+1}{2^n}) - L(r, \frac{j}{2^n}), \\ \frac{\partial}{\partial r} L(r, 0) = 0, \quad \frac{\partial}{\partial r} L(r, 1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

Above system has a unique continuous solution (Hata-Yamaguti [7], S-S [15]).

More generally we have the following:

**Theorem 2.2** (S-S [15])  $L(r, x)$  is a continuous function valued analytic function of  $r \in I$  and the equality

$$\left. \frac{\partial^k L(r, x)}{\partial r^k} \right|_{r=\frac{1}{2}} = k! T_{\frac{1}{2}, k}(x)$$

holds for  $k = 1, 2, 3, \dots$ . Here

$$\begin{aligned} T_{\frac{1}{2}, 1}(x) &= 2T(x), \\ T_{\frac{1}{2}, k}(x) &= \sum_{n=0}^{\infty} \frac{1}{2^n} R(2^n x) T_{\frac{1}{2}, k-1}(2^{n+1} x). \end{aligned}$$

**Remark 2.2** It also follows that  $T_{r, k}$  satisfies the system of infinitely many difference equations:

$$\begin{cases} T_{r, k}(\frac{2j+1}{2^{n+1}}) - (1-r) T_{r, k}(\frac{j}{2^n}) - r T_{r, k}(\frac{j+1}{2^n}) = T_{r, k-1}(\frac{j+1}{2^n}) - T_{r, k-1}(\frac{j}{2^n}), \\ T_{r, k}(0) = 0, \quad T_{r, k}(1) = 0, \\ n = 0, 1, 2, \dots, \quad j = 0, 1, \dots, 2^n - 1. \end{cases}$$

### 3 An explicit formula of exponential sums

We first give a lemma which suggests a close connection between the distribution function  $L$  and digital sums. Set  $t = \log_2 N$  for  $N \in \mathbb{N}$  and denote by  $[t]$  its integer part and by  $\{t\}$  its decimal part.

**Lemma 3.1** We have

$$(3) \quad L(r, \frac{1}{2^{1-\{t\}}}) = \sum_{n=0}^{N-1} r^{[t]+1-s(n)} (1-r)^{s(n)}.$$

Taking  $r = \frac{1}{1+e^\xi}$  in (3), we immediately have next theorem.

**Theorem 3.1** We have

$$(4) \quad F(\xi, N) = N^{\log_2(1+e^\xi)} 2^{(1-\{t\}) \log_2(1+e^\xi)} L\left(\frac{1}{1+e^\xi}, \frac{1}{2^{1-\{t\}}}\right)$$

for  $\xi \in \mathbb{R}$ .

**Remark 3.1** By (4), we know that  $G_\xi$  is differentiable for almost every  $t \in \mathbb{R}_+$ .

## 4 Asymptotic behavior of $F(\xi, N)/N^{\log_2(1+e^\xi)}$

**Theorem 4.1** *We have*

$$\limsup_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \max_{\frac{1}{2} \leq x \leq 1} x^{\log_2 r} L(r, x),$$

$$\liminf_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = \min_{\frac{1}{2} \leq x \leq 1} x^{\log_2 r} L(r, x).$$

Hence, to obtain the precise values on the left-hand sides of these equations, it suffices to estimate the function  $g(x) = x^{\log_2 r} L(r, x)$ . However, it is very hard to get the maximum and the minimum of the function  $g$ .

**Proposition 4.1** *we have*

$$\limsup_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for } \xi > 0,$$

$$\liminf_{N \rightarrow \infty} \frac{F(\xi, N)}{N^{\log_2(1+e^\xi)}} = 1 \quad \text{for } \xi < 0.$$

These estimations are obtained by Stein [16].

**Proposition 4.2** *For  $k = 1, 2, \dots, 2^{n-1} - 1, n = 1, 2, \dots$ , we have*

$$g\left(\frac{2k+1}{2^{n+1}}\right) > \min\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad \text{if } 0 < r < \frac{1}{2},$$

$$g\left(\frac{2k+1}{2^{n+1}}\right) < \max\left\{g\left(\frac{4k+1}{2^{n+2}}\right), g\left(\frac{4k+3}{2^{n+2}}\right)\right\}, \quad \text{if } \frac{1}{2} < r < 1.$$

**Remark 4.1** The above inequalities are essential in Harborth's algorithm concerned with the lower bound of the function  $F(\log 2, N)/N^{\log_2 3}$ . Harborth's algorithm is that, by starting with  $n_0 = 1$  and  $n_{r+1} = 2n_r \pm 1$  where  $+$  or  $-$  is chosen so that  $q_{r+1} = F(\log 2, n_{r+1})/n_{r+1}^{\log_2 3}$  becomes minimal. Then  $\{q_r\}$  is strictly decreasing and  $q = \lim_{n \rightarrow \infty} q_r < 0.812556 \dots$  (c.f. (2)). The question whether  $q = \lim_{n \rightarrow \infty} q_r$  gives a true lower bound is still unknown.

## 5 From exponential sums to power sums

We set

$$E(r, t) = 2^{1-\{t\}} L\left(r, \frac{1}{2^{1-\{t\}}}\right), \quad 0 < r < 1, t \in \mathbf{R}.$$

Evidently  $E(r, 0) = 2r$ ,  $E(r, 1-) = 1$ ,  $E(\frac{1}{2}, t) = 1$ , and  $E$  is continuous except for  $t \in \mathbf{Z}$  and periodic of period 1 as a function of  $t$ . Furthermore  $E$  is analytic in  $r \in (0, 1)$ . By use of  $E(r, t)$ ,

$$F(\xi, N) = (1 + e^\xi)^t \left(\frac{1 + e^\xi}{2}\right)^{1-\{t\}} E\left(\frac{1}{1 + e^\xi}, t\right).$$

On the other hand, evidently the equality

$$S_k(N) = \frac{\partial^k}{\partial \xi^k} F(\xi, N) \Big|_{\xi=0}$$

holds for  $k = 1, 2, 3, \dots$ . Hence we can directly derive explicit formulas of power sums of lower order from these equations. We set

$$E^{(k)}\left(\frac{1}{2}, t\right) = \frac{\partial^k}{\partial r^k} E(r, t) \Big|_{r=\frac{1}{2}}.$$

Then we have

$$S_1(N) = N\left(\frac{t}{2} + \frac{1-\{t\}}{2} - \frac{1}{4}E^{(1)}\left(\frac{1}{2}, t\right)\right) \quad (\text{Trollope [18], Delange [4]}),$$

$$S_2(N) = N\left(\left(\frac{t}{2}\right)^2 + H_{2,1}(t)\frac{t}{2} + H_{2,0}(t)\right) \quad (\text{Coquet [3], Osbaldestin [13]})$$

where

$$\begin{aligned} H_{2,1} &= \frac{1}{2} + 1 - \{t\} - \frac{1}{2}E^{(1)}\left(\frac{1}{2}, t\right), \\ H_{2,0} &= \frac{2 - 3\{t\} + \{t\}^2}{4} - \frac{1 - \{t\}}{4}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{1}{16}E^{(2)}\left(\frac{1}{2}, t\right), \end{aligned}$$

$$S_3(N) = N\left(\left(\frac{t}{2}\right)^3 + H_{3,2}(t)\left(\frac{t}{2}\right)^2 + H_{3,1}(t)\frac{t}{2} + H_{3,0}(t)\right)$$

(Grabner, Kirschenhofer, Prodinger and Tichy [5], O-S-S [10])

where

$$\begin{aligned} H_{3,2}(t) &= -\frac{3}{4}E^{(1)}\left(\frac{1}{2}, t\right) - \frac{3\{t\} - 6}{2}, \\ H_{3,1}(t) &= \frac{3}{16}E^{(2)}\left(\frac{1}{2}, t\right) + \frac{6\{t\} - 9}{8}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{3\{t\}^2 - 12\{t\} + 9}{4}, \\ H_{3,0}(t) &= -\frac{1}{64}E^{(3)}\left(\frac{1}{2}, t\right) - \frac{3\{t\} - 3}{32}E^{(2)}\left(\frac{1}{2}, t\right) \\ &\quad - \frac{3\{t\}^2 - 9\{t\} + 4}{16}E^{(1)}\left(\frac{1}{2}, t\right) - \frac{\{t\}^3 - 6\{t\}^2 + 9\{t\} - 4}{8}, \end{aligned}$$

$$S_4(N) = N\left(\left(\frac{t}{2}\right)^4 + H_{4,3}(t)\left(\frac{t}{2}\right)^3 + H_{4,2}(t)\left(\frac{t}{2}\right)^2 + H_{4,1}(t)\frac{t}{2} + H_{4,0}(t)\right) \quad (\text{O-S-S [11]})$$

where

$$\begin{aligned} H_{4,3}(t) &= -E^{(1)}\left(\frac{1}{2}, t\right) - 2\{t\} + 5, \\ H_{4,2}(t) &= \frac{3}{8}E^{(2)}\left(\frac{1}{2}, t\right) + \frac{3\{t\} - 6}{2}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{6\{t\}^2 - 30\{t\} + 27}{4}, \\ H_{4,1}(t) &= -\frac{1}{16}E^{(3)}\left(\frac{1}{2}, t\right) - \frac{6\{t\} - 9}{16}E^{(2)}\left(\frac{1}{2}, t\right) - \frac{3\{t\}^2 - 12\{t\} + 7}{4}E^{(1)}\left(\frac{1}{2}, t\right) \\ &\quad - \frac{2\{t\}^3 - 15\{t\}^2 + 27\{t\} - 13}{4}, \\ H_{4,0}(t) &= \frac{1}{256}E^{(4)}\left(\frac{1}{2}, t\right) + \frac{\{t\} - 1}{32}E^{(3)}\left(\frac{1}{2}, t\right) + \frac{3\{t\}^2 - 9\{t\} + 2}{32}E^{(2)}\left(\frac{1}{2}, t\right) \\ &\quad + \frac{\{t\}^3 - 6\{t\}^2 + 7\{t\} - 2}{8}E^{(1)}\left(\frac{1}{2}, t\right) + \frac{\{t\}^4 - 10\{t\}^3 + 27\{t\}^2 - 26\{t\} + 8}{16}. \end{aligned}$$

We now extend Theorem 1.1 and 1.2 and get a precise formula of  $S_p$ .

**Theorem 5.1** (O-S-S [10]) *We have*

$$S_k(N) = N \sum_{p=0}^k H_{k,p}(t) \left(\frac{t}{2}\right)^p, \quad k = 0, 1, 2, \dots$$

Here  $H_{k,p}(t)$  is a periodic continuous function of period 1, defined inductively as follows:

$$H_{0,0}(t) = E\left(\frac{1}{2}, t\right) = 1,$$

$$\begin{aligned} \frac{(-2)^k}{k!} H_{k,0}(t) &= \frac{1}{2^k k!} E^{(k)}\left(\frac{1}{2}, t\right) - \sum_{j=0}^{k-1} a(k, j, 1 - \{t\}) H_{j,0}(t), \\ \frac{(-2)^k}{k!} H_{k,p}(t) &= - \sum_{j=0}^{k-1} \sum_{q=0 \vee (p-j)}^{p \wedge (k-j)} \frac{2^q}{q!} a^{(q)}(k, j, 1 - \{t\}) H_{j,p-q}(t) \\ &\quad \text{for } p = 1, 2, \dots, k, \end{aligned}$$

where  $a(k, j, t)$  is defined by

$$\frac{2^{t-k}}{k!} \frac{\partial^k}{\partial r^k} (r^{t-s} (1-r)^s) \Big|_{r=\frac{1}{2}} = \sum_j a(k, j, t) s^j,$$

$s \in \mathbf{R}$ ,  $t \in \mathbf{R}$ ,  $k, j \in \mathbf{Z}$ ,  $k \geq 0$ , and  $a^{(p)}(k, j, t) = \partial^p a(k, j, t) / \partial t^p$ .

Furthermore the functions  $H_{k,p}(t)$  satisfies the induction formulas:

$$\sum_{j=p}^{k-1} \binom{k}{j} H_{j,p}(t) = 2^{p+1} \sum_{j=p+1}^k 2^{-j} \binom{j}{p} H_{k,j}(t)$$

for  $k \geq 1$ ,  $0 \leq p \leq k-1$ .

## References

- [1] L. E. Bush, An asymptotic formula for the average sums of the digits of integers, *Amer. Math. Monthly*, **47**(1940), 154–156.
- [2] J. Coquet, A summation formula related to the binary digits, *Invent. Math.*, **73**(1983), 107–115.
- [3] J. Coquet, Power sums of digital sums, *J. Number Theory*, **22**(1986), 161–176.
- [4] H. Delange, Sur la fonction sommatoire de la fonction "somme des chiffres", *Enseign. Math.*(2), **21**(1975), 31–47.
- [5] P. J. Grabner, P. Kirschenhofer, H. Prodinger and R. F. Tichy, On the moments of the sum-of-digits function, to appear in "Fibonacci Numbers and Applications," Vol.5, Kluwer, 1993.



- [6] H. Harborth, Number of odd binomial coefficients, *Proc. Amer. Math. Soc.* **62**(1977), 19–22.
- [7] M. Hata and M. Yamaguti, The Takagi function and its generalization, *Japan J. Appl. Math.*, **1**(1984), 183–199.
- [8] L. Mirsky, A theorem on representations of integers in the scale of  $r$ , *Scripta. Math.*, **15**(1949), 11–12.
- [9] K. Muramoto, T. Okada, T. Sekiguchi and Y. Shiota, Digital sum problems for the  $p$ -adic expansion of natural numbers, in preparation.
- [10] T. Okada, T. Sekiguchi and Y. Shiota, Application of binomial measures to power sums of digital sums, *J. Number Theory*(2), **52**(1995), 256–266.
- [11] T. Okada, T. Sekiguchi and Y. Shiota, Explicit Formulas of exponential sums of digital sums ,to appear in *Japan J. Indust. Appl. Math.*.
- [12] T. Okada, T. Sekiguchi and Y. Shiota, A Generalization of Hata-Yamaguti's results on the Takagi function II , *submitted to Japan J. Indust. Appl. Math.* .
- [13] A. H. Osbaldestin, Digital sum problems, in "Fractals in the Fundamental and Applied Sciences" Elsevier Science Publishers B. V., North-Holland, 1991.
- [14] R. Salem, On some singular monotonic functions which are strictly increasing, *Trans. Amer. Math. Soc.*, **53**(1943), 427–439.
- [15] T. Sekiguchi and Y. Shiota, A generalization of Hata-Yamaguti's results on the Takagi function, *Japan J. Indust. Appl. Math.*, **8**(1991), 203–219.
- [16] A. H. Stein, Exponential sums of sum-of-digit functions, *Illinois J. Math.*, **30**(1986), 660–675.
- [17] K. B. Stolarsky, Power and exponential sums of digital sums related to binomial coefficient parity, *SIAM J. Appl. Math.*, **32**(1977), 717–730.
- [18] J. R. Trollope, An explicit expression for binary digital sums, *Math. Mag.*, **41**(1968), 21–25.